Boundary Mean Value Property for Heat Equation

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The "heat ball" of radius $r > 0$ centered at $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ is defined as the set

$$E(x,t;r) := \left\{(y,s) \in \mathbb{R}^n \times \mathbb{R} : s \leq t, \Phi(x-y,t-s) \geq \frac{1}{r^n}\right\}$$

where Φ is the heat kernel

$$\Phi(x-y,t-s) = \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}}$$

The heat ball is of interest because it is used as the domain of integration to prove a mean value property (MVP) for (classical) solutions of the heat equation $\partial_t u - \Delta u = 0$ on some bounded open subset $U \subset \mathbb{R}^n$.

The formulation of the MVP I learned, which is from [Evans], uses the fact that volume of the unit heat ball $E(1) := E(0,0;1)$ with respect to the weight $|y|^2 / |s|^2$ is 4. The author, however, omits the details of the computation as they are an exercise in integration. Still, I do not think it is obvious to the general reader–and it certainly wasn't to me–that

$$\iint_{E(1)} \frac{|y|^2}{|s|^2} dyds = 4$$

If you have read my "About Me" section, you know that I like to work out the details, which we will do now.

The first trick to computing this integral is to write the integral as

$$\iint_{E(1)} \frac{|y|^2}{|s|^2} dyds = \int_0^{\frac{1}{4\pi}} \int_{\left\{|y|^2 \leq (2ns)\log\frac{1}{4\pi s}\right\}} \frac{|y|^2}{|s|^2} dyds$$

and recognize that the inner integral lends itself to computation in polar coordinates. Making the change of variable $y = r\omega$, where $\omega \in S^{n-1}$, we obtain

$$\begin{aligned}
\iint_{E(1)} \frac{|y|^2}{|s|^2} dyds &= \int_0^{\frac{1}{4\pi}} \int_0^{(2ns\log\frac{1}{4\pi s})^{1/2}} \int_{S^{n-1}} \frac{r^{n+1}}{|s|^2} d\omega drds \\
&= n\alpha(n) \int_0^{\frac{1}{4\pi}} \int_0^{(2ns\log\frac{1}{4\pi s})^{1/2}} \frac{r^{n+1}}{|s|^2} drds \\
&= \frac{n\alpha(n)}{n+2} \int_0^{\frac{1}{4\pi}} \frac{(2n)^{(n+2)/2}(s\log\frac{1}{4\pi s})^{(n+2)/2}}{|s|^2} ds \\
&= \frac{n\alpha(n)(2n)^{(n+2)/2}}{n+2} \int_0^{\frac{1}{4\pi}} s^{\frac{n}{2}-1} \left(\log\frac{1}{4\pi s}\right)^{\frac{n}{2}+1} ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n}{2}+1}}{(n+2)(4\pi)^{\frac{n}{2}}} \int_0^1 t^{\frac{n}{2}-1} \left(\log\frac{1}{t}\right)^{\frac{n}{2}+1} dt
\end{aligned}$$

where we make the change of variable $t = 4\pi s$ to obtain the ultimate equality. The integral factor in the above expression should remind you of the Gamma function. Indeed, the second trick is to use the identities

$$\lambda^{-z}\Gamma(z) = \int_0^1 t^{\lambda-1}\left(\log\frac{1}{t}\right)^{z-1} dt, \qquad \Gamma(z+1) = z\Gamma(z)$$

to obtain

$$\begin{aligned}
\iint_{E(1)} \frac{|y|^2}{|s|^2} dyds &= \frac{n\alpha(n)(2n)^{\frac{n}{2}+1}}{(n+2)(4\pi)^{\frac{n}{2}}} \left(\frac{n}{2}\right)^{-\frac{n}{2}-2} \Gamma\left(\frac{n}{2}+2\right) \\
&= \frac{n\alpha(n)(2n)^{\frac{n}{2}+1}}{(n+2)(4\pi)^{\frac{n}{2}}} \left(\frac{n}{2}\right)^{-\frac{n}{2}-2} \left(\frac{n}{2}+1\right) \Gamma\left(\frac{n}{2}+1\right) \\
&= \frac{2 \cdot 4^{\frac{n}{2}+1} n^{\frac{n}{2}+2} \alpha(n)(n+2)}{2(n+2)(4\pi)^{\frac{n}{2}} n^{\frac{n}{2}+2}} \Gamma\left(\frac{n}{2}+1\right) \\
&= 4 \cdot \frac{\alpha(n)}{\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}+1\right) = 4
\end{aligned}$$

Interestingly, this weighted volume is independent of the dimension n.

If you have another way of computing this integral, please share!